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## FAST TRACK COMMUNICATION

# Dynamical stability of entanglement between spin ensembles

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Online at [stacks.iop.org/JPhysA/43/072001](http://stacks.iop.org/JPhysA/43/072001)**Abstract**

We study the dynamical stability of entanglement between two spin ensembles in the presence of decoherence. For a comparative study, we consider two cases: a single spin ensemble, and two ensembles linearly coupled to a bath, respectively. In both circumstances, we assume the validity of the Markovian approximation for the bath. We examine the robustness of the state by examining the growth of the linear entropy which gives a measure of the purity of the system. We find out macroscopic entangled states of two spin ensembles can stably exist in a common bath. This result may be very useful to generate and detect macroscopic entanglement in a common noisy environment and even a stable macroscopic quantum memory.

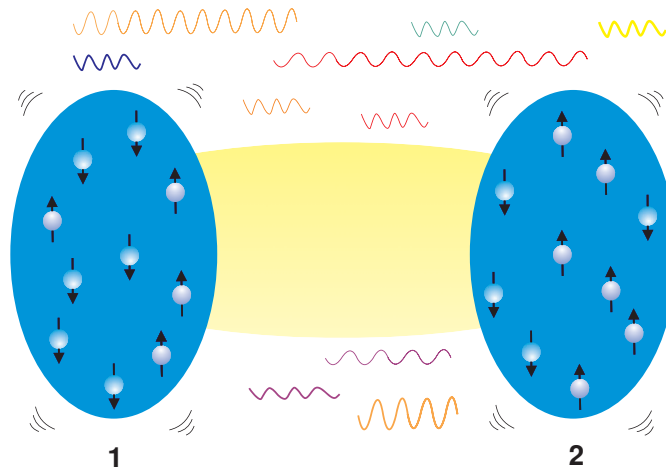
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## 1. Introduction

Quantum entanglement is a fundamental concept in quantum mechanics. It gives rise to the Einstein–Podolsky–Rosen (EPR) paradox [1] and violates a generalization of Bell’s inequality [2]. It is also the physical ingredient of quantum information processing (QIP) such as quantum communication (including quantum teleportation [3] and dense coding [4], etc). To perform quantum communication, it is required to generate entangled states in two distant locations [5]. Recently, entanglement has been generated between two separated atomic ensembles [6]. In addition, quantum interfaces between light and atoms have been experimentally demonstrated [7–9], in which the state of light was mapped onto collective excitations in atomic ensembles. This may pave the way to implement ‘long-distance’ quantum communication [10].

Atomic ensembles can be described as an ensemble of spin-half particles [10], and inevitably suffer from decoherence due to coupling to an external environment [11, 12]. This

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**Figure 1.** Two ensembles of spin-half particles are coupled to a common bath.  
(This figure is in colour only in the electronic version)

may heavily hinder the generation of entanglement and thus affect the performance of a quantum memory [10] with spin ensembles. It is thus crucial to study the robustness of the entangled states of spin ensembles under decoherence effects.

Entanglement dynamics between a pair of spin-half particles coupled to two independent baths have been studied [13, 14]. In this paper, we study two spin ensembles coupled to independent baths, as well as two spin ensembles coupled to a common bath. Two well-separated spin ensembles can be regarded as independently interacting with their respective baths. In contrast, the two spin ensembles are effectively coupled to the same bath if their separation is much shorter than the correlation length of the bath [15]. The schematic diagram is shown in figure 1. In fact, the quantum behavior for coupling to independent baths is dramatically different from the case of coupling to common baths (collective decoherence) [15, 16]. It is very important to examine the essential different features of quantum entanglement in these two decoherence models. This leads to better understanding of entanglement under decoherence and inspires us to invent useful methods to preserve the entanglement.

The effects of GHZ-type entangled states under decoherence have been studied [17]. Here we investigate the sensitivity of the bipartite entangled states to the external environment. The robustness of the states can be quantified by how long the purity of the spin ensembles can be maintained. This can be measured by means of the growth of linear entropy [12, 18, 19]. In the model of independent bath coupling, the rate of losing the purity of maximally entangled systems are found to scale with the square of the degree of entanglement.

However, we found that the entangled states of the two ensembles can exist robustly in a common bath. In particular, the singlet state can even form a decoherence-free subspace (DFS) [20]. This result shows that a macroscopic entanglement can persistently exist in a common noisy environment. It motivates the further studies of the macroscopic entanglement formation in the physical systems with the common bath. This may be useful for QIP in atom-chip based [21] and solid-state [22] systems which are required to perform short-ranged quantum communication [23, 24]. For example, two atomic Bose–Einstein condensates can be coupled to the phonon modes of an elongated condensate [25, 26] to mediate the entanglement between them.

## 2. Independent bath model

We study a decoherence model in which a spin ensemble is linearly coupled to an environment. In general, the Hamiltonian of the total system can be written as [11]

$$H = H_0 + H_B + H_I, \quad (1)$$

where  $H_0$  and  $H_B$  are the Hamiltonian of the system and the bath respectively, and  $H_I$  describes the system-bath interaction. We represent the spin ensemble in terms of the angular momentum operators:  $J = (J_x, J_y, J_z)$ . We choose the quantization axis in the  $z$ -direction such that  $J_z|m\rangle = m|m\rangle$  and  $J^2|m\rangle = j(j+1)|m\rangle$ , where  $j = N/2$  and  $N$  is the number of spin-half particles. Here we consider the subspace for  $j = N/2$  only because the atomic states are totally symmetrized for a collection of identical spin-half particles [27]. The linear interaction Hamiltonian  $H_I$ <sup>2</sup> can be expressed in a general form as [11]

$$H_I = \sum_{\alpha} J_{\alpha} \otimes B_{\alpha}, \quad (2)$$

where  $B_{\alpha}$  is the bath operator and  $\alpha = x, y$  and  $z$ .

We consider the system weakly interacting with the environment. Thus, we adopt the Markovian approximation and the master equation of the system is of the form [11]

$$\dot{\rho} = -i[H_0, \rho] + \sum_{\alpha, \beta} \gamma_{\alpha\beta} \left[ J_{\beta} \rho J_{\alpha} - \frac{1}{2} \{ J_{\alpha} J_{\beta}, \rho \} \right], \quad (3)$$

where  $\gamma_{\alpha\beta}$  is the damping constant and  $\alpha, \beta = x, y$  and  $z$ .

We can classify the decoherence model into three different cases. They are called one-, two- and three-axis models, respectively. The one-axis model can be defined as exactly one axis of the angular momentum system coupled to the bath, say  $z$ -axis. This model indeed gives a realistic description of many quantum optical phenomena [11]. The two-axis model can be defined as two axes coupled to the bath, but we presume that the coupling between the bath and one of the spin components is much stronger than the couplings of the other axes. The damping parameters can thus be written as  $\gamma_{zz} \gg \gamma_{zx}, \gamma_{xx}$ . In the three-axis model, all axes are coupled to the bath and only one of axes are strongly coupled to the bath, i.e.  $\gamma_{zz} \gg \gamma_{z\alpha}, \gamma_{\alpha\beta}$  and  $\alpha, \beta = x, y$ . The two- and three-axis models indeed provide a more general scenario for spin ensembles coupling to a bath.

To study the robustness of the states under decoherence, we can examine the stability of the growth of the linear entropy [12, 18]. The linear entropy provides a measure of the purity of a system [12]. The linear entropy can be defined as [12]

$$S_{\text{lin}} = 1 - \text{tr}(\rho^2). \quad (4)$$

A pure state gives a zero linear entropy  $S_{\text{lin}} = 0$  and  $0 \leq S_{\text{lin}} \leq 1$ . Starting with a pure state, the rate of change of the linear entropy  $\dot{S}_{\text{lin}}$  is  $-2\text{tr}(\rho\dot{\rho})$  [12].

Here we focus our investigation on the entropy production of early dynamics. This is useful for us to examine the sensitivity of the system to the environment. According to the master equation in equation (3), the rate of change of  $S_{\text{lin}}$ , with an initial pure state, can be expressed in terms of the expectation values of angular momentum operators at the time  $t = 0$ .<sup>3</sup>

$$\dot{S}_{\text{lin}}(t = 0) = 2 \sum_{\alpha, \beta} \gamma_{\alpha\beta} (\langle J_{\alpha} J_{\beta} \rangle - \langle J_{\alpha} \rangle \langle J_{\beta} \rangle). \quad (5)$$

<sup>2</sup> This model is general for the linear interaction between the system and the bath, in which all components of the angular momentum system are coupled to the bath.

<sup>3</sup> We can just substitute  $\dot{\rho}$  in equation (3) into the expression  $\dot{S}_{\text{lin}} = -2\text{tr}(\rho\dot{\rho})$ . This gives  $S_{\text{lin}} = 2 \sum_{\alpha, \beta} \text{tr}(\rho^2 J_{\alpha} J_{\beta} - \rho J_{\alpha} \rho J_{\beta})$ . Obviously, the expression in equation (5) can be obtained with an initial pure state.

Obviously, the eigenstates of  $J_z$  are in the pointer basis [28] and naturally form the DFS in the one-axis model.

Now we study the robustness of the entangled states of two spin ensembles interacting with their independent baths. We consider a general entangled state of the form

$$|\Psi_{\text{ent}}\rangle = \sum_{m=-\tilde{N}}^{\tilde{N}} c_m |m, -m\rangle, \quad (6)$$

where  $\tilde{N} = \min\{j_1, j_2\}$  and  $c_m$  is the probability coefficient. The pure-state entanglement can be quantified by the von Neumann entropy which is defined as

$$E_F = -\text{tr}(\rho_1 \ln \rho_1), \quad (7)$$

where  $\rho_1 = \text{tr}_2(\rho)$  is the reduced density matrix of  $\rho$ . Here the von Neumann entropy  $E_F$  of the state in equation (6) is  $-\sum_m |c_m|^2 \ln |c_m|^2$ . The scheme for producing this entangled state has already been proposed in the context of Bose–Einstein condensates [29, 30]. It is useful for entanglement-based quantum communications with Bose–Einstein condensates.

We consider the system starting with the entangled state in equation (6) which is an eigenstate of the system. In the one-axis model, the rate of change of the linear entropy is

$$\dot{S}_{\text{lin}} \approx 2(\gamma_{zz} + \gamma'_{zz}) \left[ \sum_m |c_m|^2 m^2 - \left( \sum_m |c_m|^2 m \right)^2 \right], \quad (8)$$

where  $\gamma_{zz}$  and  $\gamma'_{zz}$  are the two damping parameters for the two ensembles, respectively. For a nearly maximal entangled state, the probability coefficient  $|c_m|$  is roughly equal to  $1/\sqrt{2\tilde{N} + 1}$ . This gives a value of the von Neumann entropy with  $\ln |2\tilde{N} + 1|$ . We can estimate the growth of the entropy  $\dot{S}_{\text{lin}}$  which is about  $2(\gamma_{zz} + \gamma'_{zz})\tilde{N}^2/3$ .<sup>4</sup> Hence, the rate of the loss of purity scales with the square of the degree of the entanglement (the Schmidt number, i.e.  $2\tilde{N} + 1$  [31]). This means that the purity of macroscopic entanglement quickly vanishes when the two spin ensembles interact with their independent baths. However, we do not claim that the entanglement is completely lost as the purity is decreased. However, we can expect that the entanglement of formation of mixed states [32] is negligible if the state becomes highly ‘mixed’.<sup>5</sup>

### 3. Common bath model

We consider the decoherence model of the two spin ensembles linearly coupling to a common environment. The total Hamiltonian reads

$$H = H_0 + H_B + H_I, \quad (9)$$

where  $H_0$ ,  $H_B$  and  $H_I$  are the Hamiltonian of the system, the bath and the interaction between them, respectively. We represent the  $i$ th spin ensemble in terms of the usual angular momentum operators:  $J_i = (J_{ix}, J_{iy}, J_{iz})$ , where  $i = 1, 2$ . We have  $J_{iz}|m\rangle_i = m|m\rangle_i$  and  $J_i^2|m\rangle_i = j_i(j_i + 1)|m\rangle_i$ , where  $j_i = N_i/2$ .

<sup>4</sup> This quantity can be obtained by making use of the sum  $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$ . The upper bound of  $\dot{S}_{\text{lin}}$  is about  $4(\gamma_{zz} + \gamma'_{zz})\tilde{N}^3/3$ .

<sup>5</sup> The definition of entanglement of formation for mixed states is [32]:  $E_c(\rho) = \min \sum_i p_i S(\rho_i^i)$ , where  $S(\rho_i) = -\text{tr} \rho_i \ln \rho_i$  is the entropy,  $\rho_i^i = \text{tr}_2(|\psi_i\rangle\langle\psi_i|)$  is the reduced density matrix of the  $i$ th pure ensemble with the probability  $p_i$ . The probability  $p_i$  is of the order of  $j_i^{-1}$  if the state becomes highly mixed. Thus, the entropy  $E_c(\rho)$  is very small.

Without loss of generality, the interaction Hamiltonian  $H_I$  for coupling to a common bath can be written as

$$H_I = \frac{1}{2} \sum_{\alpha} [\lambda J_{1\alpha} + (2 - \lambda) J_{2\alpha}] \otimes B_{\alpha}, \quad (10)$$

where  $\lambda \in [0, 2]$  is a coupling parameter and  $B_{\alpha}$  is the bath operator and  $\alpha = x, y$  and  $z$ . The common bath model is identical to the independent bath model if we set  $\lambda = 0$  or  $2$ . We classify our decoherence model similar to the above section. We study the one-, two- and three-axis models in which the two ensembles couple to the common bath.

We consider the system and the environment in the weakly coupling regime. This enables us to adopt the Markovian approximation and write down the master equation as

$$\dot{\rho} = -i[H_0, \rho] + \sum_{\alpha, \beta} \gamma_{\alpha\beta} \left[ L_{\beta} \rho L_{\alpha} - \frac{1}{2} \{L_{\alpha} L_{\beta}, \rho\} \right], \quad (11)$$

where  $L_{\alpha} = [\lambda J_{1\alpha} + (2 - \lambda) J_{2\alpha}]/2$  is the composite angular momentum operator and  $\gamma_{\alpha\beta}$  is the damping constant. The rate of change of  $S_{\text{lin}}$ , at the time  $t = 0$ , is given by

$$\dot{S}_{\text{lin}} = 2 \sum_{\alpha, \beta} \gamma_{\alpha\beta} (\langle L_{\alpha} L_{\beta} \rangle - \langle L_{\alpha} \rangle \langle L_{\beta} \rangle). \quad (12)$$

Note that the entangled state  $|\Psi_{\text{ent}}\rangle$  in equation (6) was found to be very robust in the collective decoherence [16]. We examine the robustness of the entangled state  $|\Psi_{\text{ent}}\rangle$  in the one-axis model and this entangled state  $|\Psi_{\text{ent}}\rangle$  is an eigenstate of the system. The quantity  $\dot{S}_{\text{lin}}$  is given by

$$\dot{S}_{\text{lin}} \approx 2\gamma_{zz}(\lambda - 1)^2 \left[ \sum_m |c_m|^2 m^2 - \left( \sum_m |c_m|^2 m \right)^2 \right]. \quad (13)$$

We can estimate that  $\dot{S}_{\text{lin}}$  is about  $2\gamma_{zz}(\lambda - 1)^2 \tilde{N}^2/3$  for a highly entangled state with  $|c_m| \approx 1/\sqrt{2\tilde{N} + 1}$ . This result is consistent with the rate of the growth of entropy in the independent bath model. The losing rate of the purity also scales with the square of the degree of the entanglement for highly entangled states. However, the rate of the growth of the linear entropy can be dramatically reduced if the parameter  $\lambda$  is close to one. This is the essential feature in the collective decoherence.

In the one-axis model, the decoherence can be completely quenched as  $\lambda$  tends to one. However, the decoherence cannot be eliminated in the two-axis model even if the parameter  $\lambda$  is equal to 1. We can minimize the decoherence rate in equation (12) if its variance  $(\Delta L_x)^2$  can be kept very small for  $\lambda = 1$ . This means that the number of particles in each ensemble is nearly the same, i.e.  $j_1 \approx j_2 \approx j$ . The quantum variance  $(\Delta L_x)^2$  of the entangled state  $|\Psi_{\text{ent}}\rangle$  is given by

$$(\Delta L_x)^2 \approx \sum_m [|c_m|^2 + \text{Re}(c_{m-1}^* c_m)] [j(j+1) - m^2]. \quad (14)$$

The quantum fluctuation can be minimized if the condition is satisfied:

$$[|c_m|^2 + \text{Re}(c_{m-1}^* c_m)] \rightarrow 0. \quad (15)$$

The variance  $(\Delta L_y)^2$  is of the same form of the variance  $(\Delta L_x)^2$  and also the cross correlation  $\langle L_x L_y + L_y L_x \rangle$  is zero. Therefore, the entangled state is very stable even in the three-axis model if condition (15) can be achieved. For instance, the quantum fluctuations are greatly reduced by taking  $c_m \approx (2\tilde{N})^{-1/2}$  and  $\text{Re}(c_{m-1}^* c_m) < 0$ . Apart from that, we consider another entangled state, which also satisfies condition (15), with  $c_m \propto (-1)^m \exp[-\kappa(m \pm j')^2]$  (up to a normalization constant), where  $\kappa$  is a positive number and  $j' \approx j$ . Condition (15) can be fulfilled if  $\kappa$  is much smaller than  $1/2j'$ .

We proceed to study the robustness of eigenstates of the composite angular momentum system under the ideal collective decoherence ( $\lambda = 1$ ). In fact, the eigenstate of the composite angular momentum system in the  $z$ -direction is formed a DFS in the one-axis model. The composite eigenstate  $|L, M\rangle$  can be written as

$$|L, M\rangle = \sum_{m_{1,2}=-j_{1,2}}^{j_{1,2}} C_{j_1 m_1 j_2 m_2}^{LM} |m_1, m_2\rangle, \quad (16)$$

where  $C_{j_1 m_1 j_2 m_2}^{LM} = \langle m_1, m_2 | L, M \rangle$  is the Clebsch–Gordon coefficient and  $L = j_1 + j_2$ ,  $j_1 + j_2 - 1, \dots, |j_1 - j_2|$  and  $M = m_1 + m_2$ . The eigenstate  $|L, M = 0\rangle$  is clearly an entangled state for the two spin ensembles. The entangled pairs are formed with  $m_1 = -m_2 = m$  with the total population number  $M = 0$ .

For the state  $|L, M = 0\rangle$ , we evaluate the quantity  $\dot{S}_{\text{lin}}$  which gives  $\gamma_{xx}L(L+1)$  and  $(\gamma_{xx} + \gamma_{yy})L(L+1)$  in the two- and three-axis couplings, respectively. We can see that the state  $|L = 0, M = 0\rangle$  forms a DFS even in the three-axis linear model for  $\dot{S}_{\text{lin}} = 0$ . Indeed, this singlet state has been found to be decoherence-free [20] because it is totally symmetric to the environment. This singlet state  $|L = 0, M = 0\rangle$  gives out the maximal entanglement with the von Neumann entropy  $E_F = -\ln|2j+1|$  for  $c_m = (-1)^{j-m}/(2j+1)^{1/2}$  and  $m = -j, -j+1, \dots, j-1, j$ . Besides, the violation of the Bell inequality of large-spin system with the singlet state  $|L = 0, M = 0\rangle$  has been discussed [33].

We point out that it is difficult to produce the ideal singlet state  $|L = 0, M = 0\rangle$  in experiments because the number of particles cannot be kept to be the same in each ensemble. Nevertheless, it can be easily shown that the states  $|L, M = 0\rangle$  are also very robust in the common bath for the low values of  $L$ . This means that the states with  $M = 0$  are possible to be prepared if the number of particles in the two ensembles is very close. Rather than detecting the stable entangled state  $|\Psi_{\text{ent}}\rangle$  in equation (6) in a common noisy environment, one can also use them as quantum memory if suitable encoding and decoding mechanisms can be found.

#### 4. Discussion

We have studied the robustness of states of early dynamics in the cases of a spin ensemble and the two spin ensembles coupled to a bath, respectively. We have shown the totally different features in losing the purity in these two cases. In the independent bath model, the initial decay rate of the purity of maximally bipartite entangled states scales as the square of degree of the entanglement. This result is useful for understanding decoherence of the entanglement between two well-separated systems such as atomic ensembles coupled to two local radiation reservoirs.

In contrast, the entanglement can be preserved much longer if the two spin ensembles are coupled to a common bath. It provides a ground to detect macroscopic pure-state entanglement in a common noisy environment and perform ‘short-distance’ quantum state transmission [23, 24].

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